Chapter 1

Fourier Series

宋人有善為不龜手之藥者，世世以洴澼絖為事。客聞之，請買其方百金。聚族而謀曰： 我世世為洴澼絖，不過數金；今一朝而鬻技百金，請與之。 客得之，以說吳王。越有難，吳王使之將。冬與越人水戰，大敗越人，裂地而封之。 能不龜手，一也；或以封，或不免於洴澼絖，則所用之異也。

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In this chapter we study Fourier series. Basic definitions and examples are given in Section 1. In Section 2 we prove the fundamental Riemann-Lebesgue lemma and discuss the Fourier series from the mapping point of view. Pointwise and uniform convergence of the Fourier series of a function to the function itself under various regularity assumptions are studied in Section 3. In Section 1.5 we establish the L^2-convergence of the Fourier series without any additional regularity assumption. There are two applications. In Section 1.4 it is shown that every continuous function can be approximated by polynomials in a uniform manner. In Section 1.6 a proof of the classical isoperimetric problem for plane curves is presented.

1.1 Definition and Examples

The concept of series of functions and their pointwise and uniform convergence were discussed in Mathematical Analysis II. Power series and trigonometric series are the most important classes of series of functions. We learned power series in Mathematical Analysis II and now we discuss Fourier series.

First of all, a **trigonometric series** on $[-\pi, \pi]$ is a series of functions of the form

$$\sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx), \quad a_n, b_n \in \mathbb{R}.$$

As $\cos 0x = 1$ and $\sin 0x = 0$, we always set $b_0 = 0$ and express the series as

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

It is called a **cosine series** if all b_n vanish and **sine series** if all a_n vanish. We learned before that the most common tool in the study of the convergence of series of functions is Weierstrass M-test. Observing that

$$|a_n \cos nx + b_n \sin nx| \le |a_n| + |b_n|, \quad \forall x \in \mathbb{R},$$

we conclude from this test that the trigonometric series converges uniformly on $\mathbb{R}$ if there exist some $s > 1$ and C such that

$$|a_n| + |b_n| \le \frac{C}{n^s},$$

for all n. Denoting the limit function by f, by uniform convergence we also have

$$\begin{aligned} f(x+2\pi) &= \lim_{n\to\infty} \sum_{k=0}^{n} \big(a_k \cos(kx + 2k\pi) + b_k \sin(kx + 2k\pi)\big) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \\ &= f(x), \end{aligned}$$

hence f is a 2π-periodic, continuous function. Many delicate convergence results are available when further assumptions are imposed on the coefficients. For instance, when a_n and b_n decreasing to 0, as a consequence of the Dirichlet test, we learned in MATH2070 that the trigonometric series converges uniformly on any bounded, closed interval disjointing from the set $\{2n\pi,\ n \in \mathbb{Z}\}$. We will not go into this direction further. Here our main concern is how to represent a function in a trigonometric series.

Given a 2π-periodic function which is Riemann integrable function f on $[-\pi, \pi]$, its **Fourier series** or **Fourier expansion** is the trigonometric series given by

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy, \quad n \ge 1 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \, dy, \quad n \ge 1 \qquad \text{and} \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, dy. \end{aligned} \tag{1.1}$$

Note that a_0 is the average of the function over the interval. From this definition we gather two basic information. First, the Fourier series of a function involves the integration of the function over an interval, hence any modification of the values of the function over a subinterval, not matter how small it is, may change the Fourier coefficients a_n and b_n. This is unlike power series which only depend on the local properties (derivatives of all order at a designated point). We may say Fourier series depend on the global information but power series only depend on local information. Second, recalling from the theory of Riemann integral, we know that two integrable functions which differ at finitely many points have the same integral. Therefore, the Fourier series of two such functions are the same. In particular, the Fourier series of a function is completely determined with its value on the open interval $(-\pi, \pi)$, regardless its values at the endpoints.

The motivation of the Fourier series comes from the belief that for a "nice function" of period 2π, its Fourier series converges to the function itself. In other words, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1.2}$$

When this holds, the coefficients a_n, b_n are given by (1.1). To see this, we multiply (1.2) by $\cos mx$ and then integrate over $[-\pi, \pi]$. Using the relations

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi, & n = m \\ 0, & n \neq m \end{cases},$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad (n, m \geq 1), \qquad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx \, dx = \begin{cases} 2\pi, & n = 0 \\ 0, & n \neq 0 \end{cases},$$

we *formally* arrive at the expression of a_n, $n \geq 0$, in (1.2). Similarly, by multiplying (1.2) by $\sin mx$ and then integrate over $[-\pi, \pi]$, one obtain the expression of b_n, $n \geq 1$, in (1.2) after using

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi, & n = m \\ 0, & n \neq m \end{cases}.$$

Of course, (1.2) arises from the hypothesis that every sufficiently nice function of period 2π is equal to its Fourier expansion. The study of under which "nice conditions" this could happen is one of the main objects in the theory of Fourier series.

Trigonometric series are periodic functions of period 2π. When we compute the Fourier series of a given function which is defined only on $[-\pi, \pi]$, it is necessary to regard this function as a 2π-periodic function. For a function f defined on $(-\pi, \pi]$, the extension is straightforward. We simply let $\tilde{f}(x) = f(x - (n+1)\pi)$ where n is the unique integer satisfying $n\pi < x \leq (n+2)\pi$. It is clear that $\tilde{f}$ is equal to f on $(-\pi, \pi]$. When the function is defined on $[-\pi, \pi]$, apparently the extension is possible only if $f(-\pi) = f(\pi)$. Since the value at a point does not change the Fourier series, from now on it will be understood that the extension of a function to a 2π-periodic function refers to the extension for the restriction of this function on $(-\pi, \pi]$. Note that for a continuous function f on $[-\pi, \pi]$

its extension $\tilde{f}$ is continuous on $\mathbb{R}$ if and only if $f(-\pi) = f(\pi)$. In the following we will not distinguish f with its extension $\tilde{f}$.

We will use

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

to denote the fact that the right hand side of this expression is the Fourier series of f.

Example 1.1 We consider the function $f_1(x) = x$. Its extension is a piecewise smooth function with jump discontinuities at $(2n+1)\pi, n \in \mathbb{Z}$. As f_1 is odd and $\cos nx$ is even,

$$\pi a_n = \int_{-\pi}^{\pi} x \cos nx \, dx = 0, \quad n \geq 0,$$

and

$$\begin{aligned} \pi b_n &= \int_{-\pi}^{\pi} x \sin nx \, dx \\ &= -x \, \frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n} \, dx \\ &= (-1)^{n+1}\frac{2\pi}{n}. \end{aligned}$$

Therefore,

$$f_1(x) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Since f_1 is an odd function, it is reasonable to see that no cosine functions are involved in its Fourier series. How about the convergence of this Fourier series? Although the terms decay like $O(1/n)$ as $n \to \infty$, its convergence is not clear at this moment. On the other hand, this Fourier series is equal to 0 at $x = \pm\pi$ but $f_1(\pm\pi) = \pi$, so f_1 is not equal to its Fourier series at $\pm\pi$. It is worthwhile to observe that $\pm\pi$ are the discontinuity points of f_1.

Notation The big O and small $\circ$ notations are very convenient in analysis. We say a sequence $\{x_n\}$ satisfies $x_n = O(n^s)$ means that there exists a constant C such that $|x_n| \leq Cn^s$ as $n \to \infty$, in other words, the growth (resp. decay $s \geq 0$) of $\{x_n\}$ is not faster (resp. slower $s < 0$) the s-th power of n. On the other hand, $x_n = \circ(n^s)$ means $|x_n|n^{-s} \to 0$ as $n \to \infty$.

Example 1.2 Next consider the function $f_2(x) = x^2$. Unlike the previous example, its 2π-periodic extension is continuous on $\mathbb{R}$. After performing integration by parts, the Fourier series of f_2 is seen to be

$$f_2(x) \equiv x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx.$$

As f_2 is an even function, this is a cosine series. The rate of decay of the Fourier series is like $O(1/n^2)$. Using Weierstrass M-test (I hope you still remember it), this series converges uniformly to a continuous function. In fact, due to the following result, it converges uniformly to f_2. Note that f_2 is smooth on $(n\pi, (n+1)\pi), n \in \mathbb{Z}$.

Convergence Criterion. *The Fourier series of a continuous, 2π-periodic function which is C^1-piecewise on $[-\pi, \pi]$ converges to the function uniformly.*

A function is called C^1-**piecewise** on some interval $I = [a, b]$ if there exists a partition of I into subintervals $\{I_j\}_{j=1}^N$ and there are C^1-function f_j defined on I_j such that $f = f_j$ on each (a_j, a_{j+1}) where $I_j = [a_j, a_{j+1}]$. This convergence criterion is the special case of a theorem proved in Section 3.

We list more examples of Fourier series and leave them for you to verify.

(a) $f_3(x) \equiv |x| \sim \dfrac{\pi}{2} - \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$

(b) $f_4(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in (-\pi, 0) \end{cases} \sim \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x,$

(c) $f_5(x) = \begin{cases} x(\pi - x), & x \in [0, \pi) \\ x(\pi + x), & x \in (-\pi, 0) \end{cases} \sim \dfrac{8}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$

The convergence criterion is contained in a more general theorem we are going to prove in Section 3..

A Fourier series can also be associated to a complex-valued function. Let f be a 2π-periodic complex-valued function which is integrable on $[-\pi, \pi]$. Its Fourier series is given by the series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where the Fourier coefficients c_n are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx, \quad n \in \mathbb{Z}.$$

Here for a complex function f, its integration over some $[a, b]$ is defined to be

$$\int_a^b f(x)dx = \int_a^b f_1(x)dx + i \int_a^b f_2(x)dx,$$

where f_1 and f_2 are respectively the real and imaginary parts of f. It is called integrable if both real and imaginary parts are integrable. The expression of c_n is obtained as in the real case by first multiplying the relation

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with e^{imx} and then integrating over $[-\pi, \pi]$ with the help from the relation

$$\int_{-\pi}^{\pi} e^{imx} e^{inx}\, dx = \begin{cases} 2\pi, & n = -m \\ 0, & n \neq -m \end{cases} .$$

When f is of real-valued, we plug the relations $2\cos nx = e^{inx} + e^{-inx}$ and $2i \sin nx = e^{inx} - e^{-inx}$ into the Fourier series of f and regroup the terms. By comparing it with the Fourier series for the complex-valued function, we see that

$$c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1,$$
$$c_n = \frac{1}{2}(a_{-n} + ib_{-n}), \quad n \leq -1,$$

and $c_0 = a_0$. Note that we have $c_{-n} = \overline{c_n}$. In fact, it is easy to see that a complex Fourier series is the Fourier series of a real-valued function if and only if $c_n = \overline{c_n}$ holds for all n. The complex form of Fourier series sometimes makes expressions and computations more elegant. We will use it whenever it makes things simpler.

We have been working on the Fourier series of 2π-periodic functions. For functions of $2T$-period, their Fourier series are not the same. They can be found by a scaling argument. Let f be $2T$-periodic. The function $g(x) = f(Tx/\pi)$ is a 2π-periodic function. Thus,

$$f(Tx/\pi) = g(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $a_0, a_n, b_n, n \geq 1$ are the Fourier coefficients of g. By a change of variables, we can express everything inside the coefficients in terms of f, $\cos n\pi x/T$ and $\sin n\pi x/T$. The result is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{T} x + b_n \sin \frac{n\pi}{T} x \right),$$

where

$$\begin{aligned} a_n &= \frac{1}{T} \int_{-T}^{T} f(y) \cos \frac{n\pi}{T} y\, dy, \\ b_n &= \frac{1}{T} \int_{-T}^{T} f(y) \sin \frac{n\pi}{T} y\, dy, \quad n \geq 1, \qquad \text{and} \\ a_0 &= \frac{1}{2T} \int_{-T}^{T} f(y)\, dy. \end{aligned}$$

It reduces to (1.1) when T is equal to π.

1.2 Riemann-Lebesgue Lemma

From the examples of Fourier series in the previous section we see that the coefficients decay to 0 eventually. We will show that this is generally true. This is the content of the following result.

Theorem 1.1 (Riemann-Lebesgue Lemma). *The Fourier coefficients of a 2π-periodic function integrable on $[-\pi,\pi]$ converge to 0 as $n \to \infty$.*

We point out this theorem still holds when $[-\pi,\pi]$ is replaced by $[a,b]$. The proof is essentially the same.

We will use $R[-\pi,\pi]$ to denote the vector space of all integrable functions. To prepare for the proof we study how to approximate an integrable function by step functions. Let $a_0 = -\pi < a_1 < \cdots < a_N = \pi$ be a partition of $[-\pi,\pi]$. A **step function** s satisfies $s(x) = s_j$, $\forall x \in (a_j, a_{j+1}]$, $\forall j \geq 0$. The value of s at $-\pi$ is not important, but for definiteness let's set $s(-\pi) = s_0$. We can express a step function in a better form by introducing the characteristic function χ_E for a set $E \subset \mathbb{R}$:

$$\chi_E = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Then,

$$s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}, \quad I_j = (a_j, a_{j+1}], \quad j \geq 1, \;\; I_0 = [a_0, a_1].$$

Lemma 1.2. *For every step function in $R[-\pi,\pi]$, there exists some constant C such that*

$$|a_n|, \;\; |b_n| \leq \frac{C}{n}, \quad \forall n \geq 1,$$

where a_n, b_n are the Fourier coefficients of s.

Proof. Let $s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$. We have

$$\begin{aligned} \pi a_n &= \int_{-\pi}^{\pi} \sum_{j=0}^{N-1} s_j \chi_{I_j} \cos nx \, dx \\ &= \sum_{j=0}^{N-1} s_j \int_{a_j}^{a_{j+1}} \cos nx \, dx \\ &= \frac{1}{n} \sum_{j=0}^{N-1} s_j (\sin n a_{j+1} - \sin n a_j). \end{aligned}$$

It follows that

$$|a_n| \le \frac{C}{n}, \quad \forall n \ge 1, \qquad C = \frac{2}{\pi}\sum_{j=0}^{N-1} |s_j|.$$

Clearly a similar estimate holds for b_n. $\square$

Lemma 1.3. *Let $f \in R[-\pi,\pi]$. Given $\varepsilon > 0$, there exists a step function s such that $s \le f$ on $[-\pi,\pi]$ and*

$$\int_{-\pi}^{\pi} (f - s) < \varepsilon.$$

Proof. As f is integrable, it can be approximated from below by its Darboux lower sums. In other words, for $\varepsilon > 0$, we can find a partition $-\pi = a_0 < a_1 < \cdots < a_N = \pi$ such that

$$\left| \int_{-\pi}^{\pi} f - \sum_{j=0}^{N-1} m_j (a_{j+1} - a_j) \right| < \varepsilon,$$

where $m_j = \inf \{f(x) : x \in [a_j, a_{j+1}]\}$. It follows that

$$\left| \int_{-\pi}^{\pi} (f - s) \right| < \varepsilon$$

after setting

$$s(x) = \sum_{j=0}^{N-1} m_j \chi_{I_j}, \quad I_j = [a_j, a_{j+1}), \;\; j \ge 1, \;\; I_0 = [a_0, a_1].$$

$\square$

Now we prove Theorem 1.1.

Proof. For $\varepsilon > 0$, we can find s as constructed in Lemma 1.3 such that $0 \le f - s$ and

$$\int_{-\pi}^{\pi} (f - s) < \frac{\varepsilon}{2}.$$

Let a_n' be the n-th Fourier coefficient of s. By Lemma 1.2,

$$|a_n'| < \frac{\varepsilon}{2},$$

for all $n \ge n_0 = [2C/\varepsilon] + 1$.

$$\begin{aligned}
|\pi(a_n - a_n')| &= \left| \int_{-\pi}^{\pi} (f - s) \cos nx \, dx \right| \\
&\le \int_{-\pi}^{\pi} |f - s| \\
&= \int_{-\pi}^{\pi} (f - s) \\
&< \frac{\varepsilon}{2}\,.
\end{aligned}$$

It follows that for all $n \geq n_0$,

$$|a_n| \leq |a_n - a_n'| + |a_n'| < \frac{\varepsilon}{2\pi} + \frac{\varepsilon}{2} < \varepsilon.$$

The same argument applies to b_n too. $\square$

It is useful to bring in a "mapping" point of view between the functions and its Fourier series. Let $R_{2\pi}$ be the collection of all 2π-periodic complex-valued functions integrable on $[-\pi, \pi]$ and $\mathcal{C}$ consisting of all complex-valued bisequences $\{c_n\}$ satisfying $c_n \to 0$ as $n \to \infty$. The Fourier series sets up a mapping Φ from $R_{2\pi}$ to $\mathcal{C}$ by sending f to $\{\hat{f}(n)\}$ where, to make things clear, we have let $\hat{f}(n) = c_n$, the n-th Fourier coefficient of f. When real-functions are considered, restricting to the subspace of $\mathcal{C}$ given by those satisfying $c_n = -c_n$, Φ maps all real functions into this subspace. Perhaps the first question we ask is: Is Φ one-to-one? Clearly the answer is no, for two functions which differ on a set of measure zero have the same Fourier coefficients. However, we have the following result, to be proved in Section 5,

Uniqueness Theorem. *The Fourier series of two functions in $R_{2\pi}$ coincide if and only if they are equal except possibly at a set of measure zero.*

Thus Φ is essentially one-to-one. We may study how various structures on $R_{2\pi}$ and $\mathcal{C}$ are associated under Φ. Observe that both $R_{2\pi}$ and $\mathcal{C}$ form vector spaces over $\mathbb{C}$. In fact, there are obvious and surprising ones. Some of them are listed below and more can be found in the exercise.

Property 1. Φ is a linear map. Observe that both $R_{2\pi}$ and $\mathcal{C}$ form vector spaces over $\mathbb{R}$ or $\mathbb{C}$. The linearity of Φ is clear from its definition.

Property 2. When $f \in R_{2\pi}$ is k-th differentiable and all derivatives up to k-th order belong to $R_{2\pi}$, $\hat{f^k}(n) = (in)^k \hat{f}(n)$ for all $n \in \mathbb{Z}$. See Proposition 1.4 below for a proof. This property shows that differentiation turns into the multiplication of a factor $(in)^k$ under Φ. This is amazing!

Property 3. For $a \in \mathbb{R}$, set $f_a(x) = f(x+a)$, $x \in \mathbb{R}$. Clearly f_a belongs to $R_{2\pi}$. We have $\hat{f_a}(n) = e^{ina}\hat{f}(n)$. This property follows directly from the definition. It shows that a translation in $R_{2\pi}$ turns into the multiplication of a factor e^{ina} under Φ.

Proposition 1.4. *Let f be a differentiable, 2π-periodic function whose derivative $f' \in R_{2\pi}$. Letting*

$$f'(x) \sim c_0 + \sum_{n=1}^{\infty} \big(c_n \cos nx + d_n \sin nx\big),$$

we have

$$\begin{aligned} c_n &= nb_n \\ d_n &= -na_n, \qquad \forall n \geq 1 \end{aligned}$$

and $c_0 = 0$. In complex notations, $\hat{f'}(n) = in\hat{f}(n)$.

Proof. We compute

$$\begin{aligned}
\pi c_n &= \int_{-\pi}^{\pi} f'(y)\cos ny\,dy \\
&= f(y)\cos ny|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(y)(-n\sin ny)\,dy \\
&= n\int_{-\pi}^{\pi} f(y)\sin ny\,dy \\
&= \pi n a_n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi d_n &= \int_{-\pi}^{\pi} f'(y)\sin ny\,dy \\
&= f(y)\sin ny|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(y)n\cos ny\,dy \\
&= -n\int_{-\pi}^{\pi} f(y)\cos ny\,dy \\
&= -\pi n a_n.
\end{aligned}$$

□

Property 2 links the regularity of the function to the rate of decay of its Fourier coefficients. This is an extremely important property. When f is a 2π-periodic function whose derivatives up to k-th order belong to $R_{2\pi}$, applying Riemann-Lebesgue lemma to $f^{(k)}$ we know that $\widehat{f^{(k)}}(n) = \circ(1)$ as $n \to \infty$. By Property 2 it follows that $\hat{f}(n) = \circ(n^{-k})$, that is, the Fourier coefficients of f decay faster that n^{-k}. Since $\sum_{n=1}^{\infty} n^{-2} < \infty$, an application of M-test establishes the following result: *The Fourier series of f converges uniformly provided f, f' and f'' belong to $R_{2\pi}$.* (Be careful we cannot conclude it converges to f here. This fact will be proved in the next section.)

1.3 Convergence of Fourier Series

In this section we study the convergence of the Fourier series of a function to itself. Recall that the series $a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$, or $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where a_n, b_n, c_n are the Fourier coefficients of a function f converges to f at x means that the n-th partial sum

$$(S_n f)(x) = a_0 + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)$$

or

$$(S_n f)(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

converges to $f(x)$ as $n \to \infty$.

We start by expressing the partial sums in closed form. Indeed,

$$\begin{aligned}
(S_n f)(x) &= a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f + \sum_{k=1}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} f(y)(\cos ky \cos kx + \sin ky \sin ky)\, dy \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \Big(\frac{1}{2} + \sum_{k=1}^{n} \cos k(y-x) \Big) f(y)\, dy \\
&= \frac{1}{\pi} \int_{x-\pi}^{x+\pi} \Big(\frac{1}{2} + \sum_{k=1}^{n} \cos kz \Big) f(x+z)\, dz \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \Big(\frac{1}{2} + \sum_{k=1}^{n} \cos kz \Big) f(x+z)\, dz
\end{aligned}$$

where in the last step we have used the fact that the integrals over any two periods are the same. Using the elementary formula

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin\left(n+\frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta}{2\sin\frac{\theta}{2}}, \quad \theta \neq 0,$$

we obtain

$$(S_n f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)z}{2\sin\frac{1}{2}z} f(x+z)\, dz\ .$$

We express it in the form

$$(S_n f)(x) = \int_{-\pi}^{\pi} D_n(z) f(x+z)\, dy,$$

where D_n is the **Dirichlet kernel**

$$D_n(z) = \begin{cases} \dfrac{\sin\left(n+\frac{1}{2}\right)z}{2\pi\sin\frac{1}{2}z}, & z \neq 0 \\ \dfrac{2n+1}{2\pi}, & z = 0. \end{cases}$$

Using $\sin\theta/\theta \to 1$ as $\theta \to 0$, we see that D_n is continuous on $[-\pi, \pi]$.

Taking $f \equiv 1$, $S_n f = 1$ for all n. Hence

$$1 = \int_{-\pi}^{\pi} D_n(z)\, dz.$$

Using it we can write

$$(S_n f)(x) - f(x) = \int_{-\pi}^{\pi} D_n(z)(f(x+z) - f(x))\, dz. \tag{3.1}$$

In order to show $S_n f(x) \to f(x)$, it suffices to show the right hand side of (3.1) tends to 0 as $n \to \infty$.

Thus, the Dirichlet kernel plays a crucial role in the study of the convergence of Fourier series. We list some of its properties as follows.

Property I. $D_n(z)$ is an even, continuous, 2π-periodic function vanishing at $z = 2k\pi/(2n+1), -n \le k \le n$, on $[-\pi, \pi]$.

Property II. D_n attains its maximum value $(2n+1)/2$ at 0.

Property III.

$$\int_{-\pi}^{\pi} D_n(z)dz = 1$$

.

Property IV. For every $\delta > 0$,

$$\int_0^{\delta} |D_n(z)|dz \to \infty, \qquad \text{as } n \to \infty.$$

Only the last property needs a proof. Indeed, for each n we can fix an N such that $\pi N \le (2n+1)\delta/2 \le (N+1)\pi$, so $N \to \infty$ as $n \to \infty$. We compute

$$\begin{aligned}
\int_0^{\delta} |D_n(z)|dz &= \int_0^{\delta} \frac{|\sin(n+\frac{1}{2})z|}{2\pi|\sin\frac{z}{2}|}dz \\
&\ge \frac{1}{\pi}\int_0^{(n+\frac{1}{2})\delta} \frac{|\sin t|}{t}dt \\
&\ge \frac{1}{\pi}\int_0^{N\pi} \frac{|\sin t|}{t}dt \\
&= \frac{1}{\pi}\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}dt \\
&= \frac{1}{\pi}\sum_{k=1}^{N}\int_0^{\pi} \frac{|\sin s|}{s+(k-1)\pi}ds \\
&\ge \frac{1}{\pi}\sum_{k=1}^{N}\int_0^{\pi} \frac{|\sin s|}{\pi k}ds \\
&= c_0\sum_{k=1}^{N}\frac{1}{k}, \quad c_0 = \frac{1}{\pi^2}\int_0^{\pi} |\sin s|ds > 0, \\
&\to \infty,
\end{aligned}$$

as $N \to \infty$.

To elucidate the effect of the kernel, we fix a small $\delta > 0$ and split the integral into two parts:

$$\int_{-\pi}^{\pi} \chi_A(z) D_n(z)(f(x+z) - f(x))\, dz,$$

and

$$\int_{-\pi}^{\pi} \chi_B(z) D_n(z)(f(x+z) - f(x))\, dz,$$

where $A = (-\delta, \delta)$ and $B = [-\pi, \pi] \setminus A$. The second integral can be written as

$$\int_{-\pi}^{\pi} \frac{\chi_B(z)(f(x+z) - f(x))}{2\pi \sin \frac{z}{2}} \sin \big(n + \frac{1}{2}\big) z\, dz.$$

As $|\sin z/2|$ has a positive lower bound on B, the function

$$\frac{\chi_B(z)(f(x+z) - f(x))}{2\pi \sin \frac{z}{2}}$$

belongs to $R[-\pi, \pi]$ and the second integral tends to 0 as $n \to \infty$ in view of Riemann-Lebesgue lemma. The trouble lies on the first integral. It can be estimated by

$$\int_{-\delta}^{\delta} |D_n(z)||f(x+z) - f(x)| dz.$$

In view of Property IV, No matter how small δ is, this term may go to ∞ so it is not clear how to estimate this integral.

Rescue comes from a further regularity assumption on the function. First a definition. For a function f defined on $[a, b]$ and some $x \in [a, b]$. we call f **Lipschitz continuous** at x if there exist C and δ such that

$$|f(y) - f(x)| \leq C\, |y - x|\,, \quad \forall y \in [a, b],\ |y - x| \leq \delta. \tag{3.2}$$

Here both C and δ depend on x. The function f is called **Lipschitz continuous** on $[a, b]$ when (3.2) holds for all $x \in [a, b]$ with the same C and δ. It is better to be understood as uniformly Lipschitz continuous. Every continuously differentiable function on $[a, b]$ is Lipschitz continuous. In fact, by the fundamental theorem of calculus, for $x, y \in [a, b]$,

$$\begin{aligned} |f(y) - f(x)| &= \Big| \int_x^y f'(t) dt \Big| \\ &\leq M|y - x|, \end{aligned}$$

where $M = \sup\{|f'(t)| : \ t \in [a, b]\}$.

Theorem 1.5. *Let f be a 2π-periodic function integrable on $[-\pi, \pi]$. Suppose that f is Lipschitz continuous at x. Then $\{S_n f(x)\}$ converges to $f(x)$ as $n \to \infty$.*

Proof. Let Φ_δ be a cut-off function satisfying (a) $\Phi_\delta \in C(\mathbb{R})$, $\Phi_\delta \equiv 0$ outside $(-\delta, \delta)$, (b) $\Phi_\delta \geq 0$ and (c) $\Phi_\delta = 1$ on $(-\delta/2, \delta/2)$. We write

$$\begin{aligned}
(S_n f)(x) - f(x) &= \int_{-\pi}^{\pi} D_n(z)(f(x+z) - f(x))\,dz \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}(f(x+z)-f(x))\,dz \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_\delta(z)\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}(f(x+z)-f(x))\,dz \\
&\quad + \frac{1}{\pi}\int_{-\pi}^{\pi} (1-\Phi_\delta(z))\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}(f(x+z)-f(x))\,dz \\
&\equiv I + II\ .
\end{aligned}$$

By our assumption on f, there exists $\delta_0 > 0$ such that

$$|f(x+z) - f(x)| \leq C_0\,|z|\,, \quad \forall\,|z| < \delta_0.$$

Using $\sin\theta/\theta \to 1$ as $\theta \to 0$, there exists δ_1 such that $2|\sin z/2| \geq |z/2|$ for all $z, |z| < \delta_1$. For $z, |z| < \delta \equiv \min\{\delta_0, \delta_1\}$, we have $|f(x+z) - f(x)|/|\sin z/2| \leq 4C_0$ and

$$\begin{aligned}
|I| &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} \Phi_\delta(z)\frac{\left|\sin(n+\frac{1}{2})z\right|}{\left|\sin\frac{z}{2}\right|}\,|f(x+z)-f(x)|\;dz \\
&\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} 4C_0\,dz \\
&= \frac{4\delta C_0}{\pi}.
\end{aligned} \tag{3.3}$$

For $\varepsilon > 0$, we fix δ so that

$$\frac{4\delta C_0}{\pi} < \frac{\varepsilon}{2}. \tag{3.4}$$

After fixing δ, we turn to the second integral

$$\begin{aligned}
II &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-\Phi_\delta(z))(f(x+z)-f(x))}{\sin\frac{z}{2}}\sin\left(n+\frac{1}{2}\right)z\,dz \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-\Phi_\delta(z))(f(x+z)-f(x))}{\sin\frac{z}{2}}\left(\cos\frac{z}{2}\sin nz + \sin\frac{z}{2}\cos nz\right)\,dz \\
&\equiv \int_{-\pi}^{\pi} F_1(x,z)\sin nz\,dz + \int_{-\pi}^{\pi} F_2(x,z)\cos nz\,dz.
\end{aligned}$$

As $1-\Phi_\delta(z) = 0$, for $z \in (-\delta/2, \delta/2)$, $|\sin z/2|$ has a positive lower bound on $(-\pi, -\delta/2) \cup (\delta/2, \pi)$, and so F_1 and F_2 are integrable on $[-\pi, \pi]$. By Riemann-Lebesgue lemma, for $\varepsilon > 0$, there is some n_0 such that

$$\left|\int_{-\pi}^{\pi} F_1 \sin nz\,dz\right|,\ \left|\int_{-\pi}^{\pi} F_2 \cos nz\,dz\right| < \frac{\varepsilon}{4}, \quad \forall n \geq n_0. \tag{3.5}$$

Putting (3.3), (3.4) and (3.5) together,

$$|S_nf(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \quad \forall n \geq n_0.$$

We have shown that $S_nf(x)$ tends to $f(x)$ when f is Lipschitz continuous at x. □

We leave some remarks concerning this proof. First, the cut-off function Φ_δ can be replaced by $\chi_{[-\delta,\delta]}$ without affecting the proof. Second, the Lipschitz condition is used to kill off the growth of the kernel at x. Third, this proof is a standard one in the sense that its arguments will be used in many other places. For instance, a careful examination of it reveals a convergence result for functions with jump discontinuity after using the evenness of the Dirichlet kernel.

Theorem 1.6. *Let f be a 2π-periodic function integrable on $[-\pi, \pi]$. Suppose at some $x \in [-\pi, \pi]$, $\lim_{y\to x_+} f(y)$ and $\lim_{y\to x_-} f(y)$ exist and there are $\delta > 0$ and constant C such that*

$$|f(y) - f(x^+)| \leq C(y - x), \quad \forall y, \quad 0 < y - x < \delta,$$

and

$$|f(y) - f(x^-)| \leq C(x - y), \quad \forall y, \quad 0 < x - y < \delta.$$

Then $\{S_nf(x)\}$ converges to $(f(x^+) + f(x^-))/2$ as $n \to \infty$.

Here $f(x^+)$ and $f(x^-)$ stand for $\lim_{y\to x^+} f(y)$ and $\lim_{y\to x^-} f(y)$ respectively. We leave the proof of this theorem as an exercise.

Finally, we have

Theorem 1.7. *Let f be a Lipschitz continuous, 2π-periodic function. Then $\{S_nf\}$ converges to f uniformly as $n \to \infty$.*

Proof. Observe that when f is Lipschitz continuous on $[-\pi, \pi]$, δ_0 and δ_1 can be chosen independent of x and (3.3), (3.4) hold uniformly in x. In fact, δ_0 only depends on C_0, the constant appearing in the Lipschitz condition. Thus the theorem follows if n_0 in (3.5) can be chosen uniformly in x. This is the content of the lemma below. We apply it by taking $f(x, y)$ to be $F_1(x, z)$ or $F_2(x, z)$.

□

Lemma 1.8. *Let $f(x, y)$ be periodic in y and $f \in C([-\pi, \pi] \times [-\pi, \pi])$. For any fixed x,*

$$c(n, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x, y)e^{-iny}\, dy \to 0$$

uniformly in x as $n \to \infty$.

Proof. We need to show that for every $\varepsilon > 0$, there exists some n_0 independent of x such that

$$|c(n,x)| < \varepsilon, \quad \forall n \geq n_0.$$

Observe that

$$\begin{aligned}
2\pi c(n,x) &= \int_{-\pi}^{\pi} f(x,y)e^{-iny}\,dy \\
&= \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f\left(x, z+\frac{\pi}{n}\right) e^{-in(z+\frac{\pi}{n})}\,dz \quad y = z + \frac{\pi}{n}, \\
&= -\int_{-\pi}^{\pi} f\left(x, z+\frac{\pi}{n}\right) e^{-inz}\,dz \quad (f \text{ is } 2\pi\text{-periodic}).
\end{aligned}$$

We have

$$c(n,x) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(f(x,y) - f\left(x, y+\frac{\pi}{n}\right)\right)e^{-iny}\,dy.$$

As $f \in C([-\pi,\pi]\times[-\pi,\pi])$, it is uniformly continuous in $[-\pi,\pi]\times[-\pi,\pi]$. For $\varepsilon > 0$, there exists a δ such that

$$|f(x,y) - f(x',y')| < \varepsilon \quad \text{if } |x-x'|\,,\ |y-y'| < \delta.$$

We take n_0 so large that $\pi/n_0 < \delta$. Then, using $|e^{-iny}| = 1$,

$$\begin{aligned}
|c(n,x)| &\leq \frac{1}{4\pi}\int_{-\pi}^{\pi}\left|f(x,y) - f\left(x,y+\frac{\pi}{n}\right)\right|\,dy \\
&\leq \frac{\varepsilon}{4\pi}\int_{-\pi}^{\pi}dy = \frac{\varepsilon}{2} \\
&< \varepsilon, \qquad \forall n \geq n_0.
\end{aligned}$$

□

Example 1.3. We return to the functions discussed in Examples 1.1 and 1.2 Indeed, f_1 is smooth except at $n\pi$. According to Theorem 1.5, the series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

converges to x for every $x \in (-\pi,\pi)$. On the other hand, we observed before that the series tend to 0 at $x = \pm\pi$. As $f_1(\pi_+) = -\pi$ and $f(\pi_-) = \pi$, we have $f_1(\pi_+) + f(\pi_-) = 0$, which is in consistency with Theorem 1.5. In the second example, f_2 is continuous, 2π-periodic. By Theorem 1.7, its Fourier series

$$\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

converges to x^2 uniformly on $[-\pi, \pi]$.

Theorems 1.5–1.7 in fact applies to a larger class of continuous functions. A function f on $[a, b]$ is called **Hölder continuous** at $x \in [a, b]$ if there exist $\alpha \in (0, 1)$, a constant C, and δ such that

$$|f(y) - f(x)| \leq C\,|y - x|^{\alpha}, \quad \forall y \in [a, b],\ |y - x| \leq \delta. \tag{3.6}$$

The number α is called the **Hölder exponent** of f. A function is **Hölder continuous** on $[a, b]$ if (3.6) holds for all $x \in [a, b]$. We will denote the collection of all Hölder continuous, 2π-periodic functions with Hölder exponent α by $\mathcal{C}^{0,\alpha}_{2\pi}$, $\alpha \in (0, 1)$. Also, denote $\mathcal{C}^{0,1}_{2\pi}$ the collection of all Lipschitz continuous, 2π-periodic functions. These are vector spaces. A straightforward modification the proofs in Theorems 1.5, 1.6 and 1.7 shows that they still hold when the Lipschitz continuity condition is replaced by a Hölder continuity condition.

If we let $\mathcal{C}^{k}_{2\pi}$ be the vector space of all k-times continuously differentiable, 2π-periodic functions, we have the following scale of regularity

$$\mathcal{C}^{0}_{2\pi} \subset \mathcal{C}^{0,\alpha}_{2\pi} \subset \mathcal{C}^{0,\beta}_{2\pi} \subset \cdots \subset \mathcal{C}^{1}_{2\pi} \subset \mathcal{C}^{2}_{2\pi} \subset \cdots, \quad \alpha < \beta \leq 1.$$

(Very often we write $\mathcal{C}_{2\pi}$ instead of $\mathcal{C}^{0}_{2\pi}$.) It is not hard to show that all these inclusions are proper. This scale of regularity of functions is useful in many occasions.

So far we have been working on the Fourier series of 2π-periodic functions. It is clear that the same results apply to the Fourier series of $2T$-periodic functions for arbitrary positive T.

We have shown the convergence of the Fourier series under some additional regularity assumptions on the function. But the basic question remains, that is, is the Fourier series of a continuous, 2π-periodic function converges to itself? It turns out the answer is negative. An example can be found in Stein-Shakarchi. In fact, using the uniform boundedness principle in functional analysis, one can even show that most continuous functions have divergent Fourier series. The situation is very much like in the case of the real number system where transcendental numbers are uncountable while algebraic numbers are countable despite the fact that it is difficult to establish a specific number is transcendental.

Theorem 1.9 and Proposition 1.10 are for optional reading.

We present another convergence result where is concerned with pointwise convergence. It replaces regularity by monotonicity in the function under consideration.

Theorem 1.9. *Let f be a 2π-periodic function integrable on $[-\pi, \pi]$. Suppose that it is piecewise continuous and increasing near some point x. Its Fourier series converges to $\big(f(x^+) + f(x^-)\big)/2$ at x.*

Proof. In the following proof we will take $x = 0$ for simplicity. We first write, using the evenness of D_n,

$$\begin{aligned}(S_n f)(0) &= \int_{-\pi}^{\pi} D_n(z) f(z)\, dz \\ &= \int_0^{\pi} (f(z) + f(-z)) D_n(z)\, dz.\end{aligned}$$

So,

$$(S_n f)(0) - \frac{1}{2}(f(0^+) + f(0^-)) = \int_0^{\pi} (f(z) - f(0^+) + f(-z) - f(0^-)) D_n(z)\, dz.$$

We will show that

$$\int_0^{\pi} (f(z) - f(0^+)) D_n(z)\, dz \to 0 \tag{3.7}$$

and

$$\int_0^{\pi} (f(-z) - f(0^-)) D_n(z)\, dz \to 0 \tag{3.8}$$

as $n \to \infty$. Indeed, for a small $h > 0$, we consider

$$\begin{aligned}\int_0^h (f(z) - f(0^+)) \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}}\, dz &= \int_0^h (f(z) - f(0^+)) \frac{\sin \frac{2n+1}{2} z}{\frac{z}{2}}\, dz \\ &\quad + \int_0^h (f(z) - f(0^+)) \left(\frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}} - \frac{\sin \frac{2n+1}{2} z}{\frac{z}{2}} \right) dz.\end{aligned}$$

Using L'Hospital's rule,

$$\frac{1}{\sin \frac{z}{2}} - \frac{1}{\frac{z}{2}} = \frac{z - 2\sin \frac{z}{2}}{z \sin \frac{z}{2}} \to 0 \quad \text{as } z \to 0.$$

Therefore, for $\varepsilon > 0$, we can find h_1 such that

$$\begin{aligned}&\int_0^{h_1} \left| f(z) - f(0^+) \right| \left| \frac{1}{\sin \frac{z}{2}} - \frac{1}{\frac{z}{2}} \right| \left| \sin \frac{2n+1}{2} z \right| dz \\ \leq &\int_0^{h_1} \left| f(z) - f(0^+) \right| \left| \frac{1}{\sin \frac{z}{2}} - \frac{1}{\frac{z}{2}} \right| dz < \frac{\varepsilon}{3},\end{aligned}$$

where h_1 is independent of n. Next, by the second mean-value theorem for integral (see below),

$$\int_0^h (f(z) - f(0^+)) \frac{\sin \frac{2n+1}{2} z}{\frac{z}{2}}\, dz = (f(h) - f(0^+)) \int_k^h \frac{\sin \frac{2n+1}{2} z}{\frac{z}{2}}\, dz$$

for some $k \in (0,h)$. As

$$\begin{aligned}\left|\int_k^h \frac{\sin\frac{2n+1}{2}z}{\frac{z}{2}}\,dz\right| &= \left|2\int_{\ell k}^{\ell h}\frac{\sin t}{t}\,dt\right|, \quad \ell = \frac{2n+1}{2}\\ &\le 2\left|\int_0^{\ell h}\frac{\sin t}{t}\,dt\right| + 2\left|\int_0^{\ell k}\frac{\sin t}{t}\,dt\right|\\ &\le 4\sup_T\left|\int_0^T\frac{\sin t}{t}\,dt\right| \equiv 4L,\end{aligned}$$

and we can find $h_2 \le h_1$ such that

$$4L\left|f(h) - f(0^+)\right| < \frac{\varepsilon}{3}, \quad \forall 0 < h \le h_2,$$

we have

$$\left|\int_0^{h_2}(f(z) - f(0^+))\frac{\sin\frac{2n+1}{2}z}{\sin\frac{z}{2}}\,dz\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Now, by Riemann-Lebesgue lemma, there exists some n_0 such that

$$\left|\int_{h_2}^{\pi}(f(z) - f(0^+))\frac{\sin\frac{2n+1}{2}z}{\sin\frac{z}{2}}\,dz\right| < \frac{\varepsilon}{3}, \quad \forall n \ge n_0.$$

Putting things together,

$$\left|\int_0^{\pi}(f(z) - f(0^+))\frac{\sin\frac{2n+1}{2}z}{\sin\frac{z}{2}}\,dz\right| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \forall n \ge n_0.$$

We have shown that (3.7) holds. To prove (3.8), it suffices to apply (3.7) to the function $g(z) = f(-z)$.

□

Proposition 1.10 (Second Mean-Value Theorem). *Let $f \in R[a,b]$ and g be monotone on $[a,b]$ and satisfy $g(a) = 0$. There exists some $c \in (a,b)$ such that*

$$\int_a^b f(x)g(x)\,dx = g(b)\int_c^b f(x)\,dx.$$

Proof. Without loss of generality, we assume g is increasing. Let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition P on $[a,b]$.

$$\int_a^b fg = \sum_{j=1}^n g(x_j)\int_{x_{j-1}}^{x_j} f + \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)(g(x) - g(x_j))\,dy.$$

In case $\|P\| \to 0$, it is not hard to show that the second integral tends to zero, so

$$\int_a^b fg = \lim_{\|P\|\to 0} \sum_{j=1}^n g(x_j) \int_{x_{j-1}}^{x_j} f.$$

Letting $F(x) = \int_x^b f$ and using $F(x_n) = F(b) = 0$, we have

$$\begin{aligned}\sum_{j=1}^n g(x_j) \int_{x_{j-1}}^{x_j} f &= \sum_{j=1}^n g(x_j)(F(x_j) - F(x_{j-1})) \\ &= g(x_1)F(x_0) + \sum_{j=1}^{n-1} (g(x_{j+1}) - g(x_j))F(x_j).\end{aligned}$$

Let $m = \inf_{[a,b]} F$ and $M = \sup_{[a,b]} F$. As g is increasing,

$$mg(b) \le g(x_1)F(x_0) + \sum_{j=1}^{n-1} (g(x_{j+1}) - g(x_j))F(x_j) \le Mg(b).$$

Letting $\|P\| \to 0$, we conclude that

$$mg(b) \le \int_a^b fg \le Mg(b).$$

As $c \mapsto \int_c^b f$ is continuous and bounded between m and M, there is some c such that

$$\frac{1}{g(b)} \int_a^b fg = \int_c^b f.$$

□

1.4 Weierstrass Approximation Theorem

As an application of Theorem 1.7, we prove a theorem of Weierstrass concerning the approximation of continuous functions by polynomials. First we consider how to approximate a continuous function by continuous, piecewise linear functions. A continuous function defined on $[a, b]$ is **piecewise linear** if there exists a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that f is linear on each subinterval $[a_j, a_{j-1}]$.

Proposition 1.11. *Let f be a continuous function on $[a, b]$. For every $\varepsilon > 0$, there exists a continuous, piecewise linear function g such that $\|f - g\|_\infty < \varepsilon$.*

Recall that $\|f - g\|_\infty = \sup\{|f(x) - g(x)| : x \in [a, b]\}$.

Proof. As f is uniformly continuous on $[a,b]$, for every $\varepsilon > 0$, there exists some δ such that $|f(x) - f(y)| < \varepsilon/2$ for $x, y \in [a,b], |x-y| < \delta$. We partition $[a,b]$ into subintervals $I_j = [a_j, a_{j+1}]$ whose length is less than δ and define g to be the piecewise linear function satisfying $g(a_j) = f(a_j)$ for all j. For $x \in [a_j, a_{j+1}]$, g is given by

$$g(x) = \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) + f(a_j).$$

We have

$$\begin{aligned} |f(x) - g(x)| &= |f(x) - \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) + f(a_j)| \\ &\le |f(x) - f(a_j)| + |\frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j)| \\ &\le |f(x) - f(a_j)| + |f(a_{j+1}) - f(a_j)| \\ &< \varepsilon, \end{aligned}$$

the result follows. □

Next we study how to approximate a continuous function by trigonometric polynomials (or, equivalently, finite Fourier series).

Proposition 1.12. *Let f be a continuous function on $[0,\pi]$. For $\varepsilon > 0$, there exists a trigonometric polynomial h such that $\|f - h\|_\infty < \varepsilon$.*

Proof. First we extend f to $[-\pi,\pi]$ by setting $f(x) = f(-x)$ (using the same notation) to obtain a continuous function on $[-\pi,\pi]$ with $f(-\pi) = f(\pi)$. By the previous proposition, we can find a continuous, piecewise linear function g such that $\|f - g\|_\infty < \varepsilon/2$. Since $g(-\pi) = f(-\pi) = f(\pi) = g(\pi)$, g can be extended as a Lipschitz continuous, 2π-periodic function. By Theorem 1.7, there exists some N such that $\|g - S_N g\|_\infty < \varepsilon/2$. Therefore, $\|f - S_N g\|_\infty \le \|f - g\|_\infty + \|g - S_N g\|_\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon$. The proposition follows after noting that every finite Fourier series is a trigonometric polynomial (see Exercise 1). □

Theorem 1.13 (Weierstrass Approximation Theorem). *Let $f \in C[a,b]$. Given $\varepsilon > 0$, there exists a polynomial p such that $\|f - p\|_\infty < \varepsilon$.*

Proof. Consider $[a,b] = [0,\pi]$ first. Extend f to $[-\pi,\pi]$ as before and, for $\varepsilon > 0$, fix a trigonometric polynomial h such that $\|f - h\|_\infty < \varepsilon/2$. This is possible due to the previous proposition. Now, we express h as a finite Fourier series $a_0 + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx)$. Using the fact that

$$\cos\theta = \sum_{n=0}^\infty \frac{(-1)^n \theta^{2n}}{(2n)!}, \quad \text{and } \sin\theta = \sum_{n=1}^\infty \frac{(-1)^{n-1}\theta^{2n-1}}{(2n-1)!},$$

where the convergence is uniform on $[-\pi, \pi]$, each $\cos nx$ and $\sin nx, n = 1, \cdots, N$, can be approximated by polynomials. Putting all these polynomials together we obtain a polynomial $p(x)$ satisfying $\|h - p\|_\infty < \varepsilon/2$. It follows that $\|f - p\|_\infty \le \|f - h\|_\infty + \|h - p\|_\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

When f is continuous on $[a, b]$, the function $\varphi(t) = f(\frac{b-a}{\pi}t + a)$ is continuous on $[0, \pi]$. From the last paragraph, we can find a polynomial $p(t)$ such that $\|\varphi - p\|_\infty < \varepsilon$ on $[0, \pi]$. But then the polynomial $q(x) = p(\frac{\pi}{b-a}(x - a))$ satisfies $\|f - q\|_\infty = \|\varphi - p\|_\infty < \varepsilon$ on $[a, b]$. $\square$

1.5 Mean Convergence of Fourier Series

In Section 2 we studied the uniform convergence of Fourier series. Since the limit of a uniformly convergent series of continuous functions is again continuous, we do not expect results like Theorem 1.6 applies to functions with jumps. In this section we will measure the distance between functions by a measurement weaker than the uniform norm. Under the new L^2-distance, every integrable function is equal to its Fourier expansion.

Recall that there is an inner product defined on the n-dimensional Euclidean space called the Euclidean metric

$$\langle x, y \rangle_2 = \sum_{j=1}^{n} x_j y_j, \qquad x, y \in \mathbb{R}^n.$$

With this inner product, one can define the concept of orthogonality and angle between two vectors. Likewise, we can also introduce a similar product on the space of integrable functions. Specifically, for $f, g \in R[-\pi, \pi]$, the L^2**-product** is given by

$$\langle f, g \rangle_2 = \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

The L^2-product behaves like the Euclidean metric on $\mathbb{R}^n$ except at one point, namely, the condition $\langle f, f \rangle_2 = 0$ does not imply $f \equiv 0$. This is easy to see. In fact, when f is equal to zero except at finitely many points, then $\langle f, f \rangle_2 = 0$. A result of Lebesgue which characterizes this situation asserts that $\langle f, f \rangle_2 = 0$ if and only if f is equal to zero except on a set of measure zero. This minor difference with the Euclidean inner product will not affect our discussion much. Parallel to the Euclidean case, we define the L^2-norm of an integrable function f to be

$$\|f\|_2 = \sqrt{\langle f, f \rangle_2},$$

and the L^2**-distance** between two integrable functions f and g by $\|f - g\|_2$. Then we can talk about $f_n \to f$ in L^2-sense, i.e.,

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |f - f_n|^2 = 0.$$

This is a convergence in an average sense. It is not hard to see that when $\{f_n\}$ tends to f uniformly, $\{f_n\}$ must tend to f in L^2-sense. A moment's reflection will show that the converse is not always true. Hence convergence in L^2-sense is weaker than uniform convergence. Our aim is to show that the Fourier series of every integrable function converges to the function in L^2-sense.

Just like the canonical basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$, the functions

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty}$$

forms an orthonormal basis in $R[-\pi, \pi]$, see Section 1.1. In the following we denote by

$$E_n = \left\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle_{j=1}^{n}$$

the $(2n+1)$-dimensional vector space spanned by the first $2n+1$ trigonometric functions.

We start with a general result. Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal set (or orthonormal family) in $R[-\pi, \pi]$, i.e.,

$$\int_{-\pi}^{\pi} \phi_n \phi_m = \delta_{nm}, \quad \forall n, m \geq 1.$$

Let

$$\mathcal{S}_n = \langle \phi_1, \ldots, \phi_n \rangle$$

be the n-dimensional subspace spanned by $\phi_1, \ldots, \phi_n$. For a general $f \in R[-\pi, \pi]$, we consider the minimization problem

$$\inf \left\{ \|f - g\|_2 : g \in \mathcal{S}_n \right\}. \tag{5.1}$$

From a geometric point of view, this infimum gives the L^2-distance from f to the finite dimensional subspace $\mathcal{S}_n$.

Proposition 1.14. *The unique minimizer of (5.1) is attained at the function* $g = \sum_{j=1}^{n} \alpha_j \phi_j$*, where* $\alpha_j = \langle f, \phi_j \rangle$*.*

Proof. To minimize $\|f - g\|_2$ is the same as to minimize $\|f - g\|_2^2$. Every g in $\mathcal{S}_n$ can be written as $g = \sum_{j=1}^{n} \beta_j \phi_j$, $\beta_j \in \mathbb{R}$. Let

$$\begin{aligned}
\Phi(\beta_1, \ldots, \beta_n) &= \int_{-\pi}^{\pi} |f - g|^2 \\
&= \int_{-\pi}^{\pi} \Big(f - \sum_{j=1}^{n} \beta_j \phi_j \Big)^2 \\
&= \int_{-\pi}^{\pi} f^2 - 2 \sum_{j=1}^{n} \beta_j \alpha_j + \sum_{j=1}^{n} \beta_j^2 .
\end{aligned}$$

be a function from $\mathbb{R}^n$ to $\mathbb{R}$. We use elementary inequality $2ab \le a^2 + b^2$ in a tricky way,

$$\begin{aligned} 2\sum_{j=1}^n \beta_j\alpha_j &= 2\sum_{j=1}^n \frac{\beta_j}{\sqrt{2}}\sqrt{2}\alpha_j \\ &\le \sum_{j=1}^n \frac{\beta_j^2}{2} + 2\sum_{j=1}^n \alpha_j^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(\beta) &\ge \int_{-\pi}^{\pi} f^2 - \frac{1}{2}\sum_{j=1}^n \beta_j^2 - 2\sum_{j=1}^n \alpha_j^2 + \sum_{j=1}^n \beta_j^2 \\ &= \int_{-\pi}^{\pi} f^2 - 2\sum_{j=1}^n \alpha_j^2 + \frac{1}{2}|\beta|^2 \\ &\to \infty, \end{aligned}$$

as $|\beta| \to \infty$. It implies that Φ must attain a minimum at some finite point γ. At this point γ, $\nabla\Phi(\gamma) = (0, \dots, 0)$. We compute

$$\frac{\partial \Phi}{\partial \beta_i} = -2\alpha_i + 2\beta_i.$$

Hence, $\beta = \alpha$. As there is only one critical point, it must be the minimum of Φ. □

Given an orthonormal family $\{\phi_n\}_{n=1}^\infty$, one may define the *Fourier series* of an L^2-function f *with respect to* the orthnormal family $\{\phi_n\}$ to be the series $\sum_{n=1}^\infty \langle f, \phi_n\rangle \phi_n$ and set $S_n f = \sum_{k=1}^n \langle f, \phi_k\rangle \phi_k$ as before. This proposition asserts that the distance between f and $\mathcal{S}_n$ is realized at $\|f - S_n f\|_2$. The function $S_n f$ is sometimes called the orthogonal projection of f on $\mathcal{S}_n$. As a special case, taking $\{\phi_n\} = \left\{1/\sqrt{2\pi}, \cos nx/\sqrt{\pi}, \sin nx/\sqrt{\pi}\right\}$ and $\mathcal{S}_{2n+1} = E_n$, we get

Corollary 1.15. *For $f \in R[-\pi, \pi]$, for each $n \ge 1$,*

$$\|f - S_n f\|_2 \le \|f - g\|_2$$

for all g of the form

$$g = c_0 + \sum_{k=1}^n (c_j \cos kx + d_j \sin kx), \quad c_0, c_k, d_k \in \mathbb{R}.$$

Here is the main result of this section.

Theorem 1.16. *For every $f \in R[-\pi, \pi]$,*

$$\lim_{n\to\infty} \|S_n f - f\|_2 = 0.$$

Proof. Let $f \in R[-\pi, \pi]$. For $\varepsilon > 0$, we can find a 2π-periodic, Lipschitz continuous function g such that
$$\|f - g\|_2 < \frac{\varepsilon}{2}.$$
Indeed, g can be obtained by first approximating f by a step function and then modifying the step function at its jumps. By Theorem 1.7, we can fix an N so that in sup-norm
$$\|g - S_N g\|_\infty < \frac{\varepsilon}{2\sqrt{2\pi}}.$$
Thus
$$\|g - S_N g\|_2 = \sqrt{\int_{-\pi}^{\pi} (g - S_N g)^2} \le \|g - S_N g\|_\infty \sqrt{2\pi} < \frac{\varepsilon}{2}.$$
It follows from Corollary 1.15 that
$$\begin{aligned} \|f - S_N f\|_2 &\le \|f - S_N g\|_2 \\ &\le \|f - g\|_2 + \|g - S_N g\|_2 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$
As $\mathcal{S}_N \subset \mathcal{S}_n$ for all $n \ge N$, by Corollary 1.15 again, we have
$$\|f - S_n f\|_2 \le \|f - S_N f\|_2 < \varepsilon.$$
□

We have the following result concerning the uniqueness of the Fourier expansion.

Corollary 1.17. *(a) Suppose that f_1 and f_2 in $R_{2\pi}$ have the same Fourier series. Then f_1 and f_2 are equal almost everywhere.*

(b) Suppose that f_1 and f_2 in $\mathcal{C}_{2\pi}$ have the same Fourier series. Then $f_1 \equiv f_2$.

Proof. Let $f = f_2 - f_1$. The Fourier coefficients of f all vanish, hence $S_n f = 0$, for all n. By Theorem 1.16, $\|f\|_2 = 0$. From the theory of Riemann integral we know that f^2, hence f, must vanish except on a set of measure zero. In other words, f_2 is equal to f_1 almost everywhere. (a) holds. To prove (b), letting f be continuous and assuming $f(x)$ is not equal to zero at some x, by continuity it is non-zero for all points near x. Hence we may assume x belongs to $(-\pi, \pi)$ and $|f(y)| > 0$ for all $y \in (x - \delta, x + \delta)$ for some $\delta > 0$. But then $\|f\|_2$ would be greater or equal to the integral of $|f|$ over $(x - \delta, x + \delta)$, which is positive. This contradiction shows that $f \equiv 0$. □

Another interesting consequence of Theorem 1.16 is the Parseval's identity. In fact, this identity is equivalent to Theorem 1.16.

Corollary 1.18 (Parseval's Identity). *For every $f \in R[-\pi, \pi]$,*

$$\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right),$$

where a_n and b_n are the Fourier coefficients of f.

Proof. Making use of the relations such as $\langle f, \cos nx/\sqrt{\pi}\rangle_2 = \sqrt{\pi}a_n, n \geq 1$, we have $\langle f, S_n f\rangle_2 = \|S_n f\|_2^2 = 2\pi a_0^2 + \pi \sum_{k=1}^{n}(a_k^2 + b_k^2)$. By Theorem 1.15,

$$\begin{aligned} 0 = \lim_{n\to\infty} \|f - S_n f\|_2^2 &= \lim_{n\to\infty} \left(\|f\|_2^2 - 2\langle f, S_n f\rangle_2 + \|S_n f\|_2^2\right) \\ &= \lim_{n\to\infty} \left(\|f\|_2^2 - \|S_n f\|_2^2\right) \\ &= \|f\|_2^2 - \left[2\pi a_0^2 + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)\right]. \end{aligned}$$

□

The norm of f can be regarded as the length of the "vector" f. Parseval's identity shows that the square of the length of f is equal to the sum of the square of the length of the orthogonal projection of f onto each one-dimensional subspace spanned by the sine and cosine functions. This is an infinite dimensional version of the ancient Pythagoras theorem. It is curious to see what really comes out when you plug in some specific functions. For instance, we take $f(x) = x$ and recall that its Fourier series is given by $\sum 2(-1)^{n+1}/n \sin nx$. Therefore, $a_n = 0, n \geq 0$ and $b_n = 2(-1)^{n+1}\sqrt{\pi}/n$ and Parseval's identity yields Euler's summation formula

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

You could find more interesting identities by applying the same idea to other functions.

The following result will be used in the next section.

Corollary 1.19 (Poincaré's Inequality). *For every $f \in \mathcal{C}_{2\pi}^1$,*

$$\int_{-\pi}^{\pi} (f(x) - \overline{f})^2 dx \leq \int_{-\pi}^{\pi} f'^2(x) dx \ ,$$

and equality holds if and only if $a_n = b_n = 0$ for all $n \geq 2$.

Here

$$\overline{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f$$

is the average or mean of f over $[-\pi, \pi]$.

Proof. Noting that

$$\overline{f} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos 0x dx = a_0,$$

$$f(x) - \overline{f} = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

by Theorem 1.6. By Parseval's identity,

$$\int_{-\pi}^{\pi}(f(x) - \overline{f})^2 dx = \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\ ,$$

and

$$\int_{-\pi}^{\pi}(f(x) - \overline{f})'^2 dx = \int_{-\pi}^{\pi} f'(x)^2 dx = \pi\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2).$$

Therefore, we have

$$\int_{-\pi}^{\pi} f'(x)^2 dx - \int_{-\pi}^{\pi}(f(x) - \overline{f})^2 dx = \pi\sum_{n=1}^{\infty}(n^2 - 1)(a_n^2 + b_n^2)\ ,$$

and the result follows.

□

This inequality is also known as Wirtinger's inequality.

1.6 The Isoperimetric Problem

The classical isoperimetric problem known to the ancient Greeks asserts that only the circle maximizes the enclosed area among all simple, closed curves of the same perimeter. In this section we will present a proof of this inequality by Fourier series. To formulate this geometric problem in analytic terms, we need to recall some facts from advanced calculus.

Indeed, a parametric C^1-curve is a map γ from some interval $[a, b]$ to $\mathbb{R}^2$ such that x and y belong to $C^1[a, b]$ where $\gamma(t) = (x(t), y(t))$ and $x'^2(t) + y'^2(t) > 0$ for all $t \in [a, b]$. In the following a curve is always refereed to a parametric C^1-curve. For such a curve, its length is defined to be

$$L[\gamma] = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt, \quad \gamma = (x, y).$$

A curve is closed if $\gamma(a) = \gamma(b)$ and simple if $\gamma(t) \neq \gamma(s)$, $\forall t \neq s$ in $[a, b)$. The length of a closed curve is called the perimeter of the curve.

When a closed, simple curve is given, the area it encloses is also fixed. Hence one should be able to express this enclosed area by a formula involving γ only. Indeed, this can be accomplished by the Green's theorem. Recalling that the Green's theorem states that for every pair of C^1-functions P and Q defined on the curve γ and the region enclosed by the curve, we have

$$\int_\gamma Pdx + Qdy = \iint_D \Big(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\Big)dxdy \ ,$$

where the left hand side is the line integral along γ and D is the domain enclosed by γ (see Fritzpatrick, p.543). Taking $P \equiv 0$ and $Q = x$, we obtain

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy = \iint_D 1 = \text{ area of } D,$$

so

$$A[\gamma] = \iint_D 1dxdy = \int_\gamma xdy = \int_0^1 x(t)y'(t)dt \ .$$

The classical isoperimetric problem is: Among all simple, closed curves with a fixed perimeter, find the one whose enclosed area is the largest. We will see that the circle is the only solution to this problem.

To proceed further, let us recall the concept of reparametrization. Indeed, a curve γ_1 on $[a_1, b_1]$ is called a reparametrization of the curve γ on $[a, b]$ if there exists a C^1-map ξ from $[a_1, b_1]$ to $[a, b]$ with non-vanishing derivative so that $\gamma_1(t) = \gamma(\xi(t)), \ \forall t \in [a_1, b_1]$. It is known that the length remains invariant under reparametrizations.

Another useful concept is the parametrization by arc-length. A curve $\gamma = (x, y)$ on $[a, b]$ is called in arc-length parametrization if $x^{'2}(t) + y^{'2}(t) = 1, \ \forall t \in [a, b]$. We know that every curve can be reparametrized in arc-length parametrization. Let $\gamma(t) = (x(t), y(t)), \ t \in [a, b]$, be a parametrization of a curve. We define a function φ by setting

$$\varphi(z) = \int_a^z \big(x^{'2}(t) + y^{'2}(t)\big)^{1/2} dt,$$

it is readily checked that φ is a C^1-map from $[a, b]$ to $[0, L]$ with positive derivative, and $\gamma_1(s) = \gamma(\xi(s)), \ \xi = \varphi^{-1}$, is an arc-length reparametrization of γ on $[0, L]$ where L is the length of γ.

We now apply the Poincaré's inequality to give a proof of the classical isoperimetric problem.

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed, simple C^1-curve bounding a region D. Without loss of generality we may assume that it is parametrized by arc-length. Assuming the perimeter of γ is equal to 2π, we want to find the region that encloses the maximal area. The perimeter is given by

$$L[\gamma] = \int_0^{2\pi} \sqrt{x'^2(s) + y'^2(s)} ds = 2\pi \ ,$$

and the area is given by

$$A[\gamma] = \int_0^{2\pi} x(s)y'(s)ds \ .$$

We compute

$$\begin{aligned}
2A[\gamma] &= \int_{-\pi}^{\pi} 2x(s)y'(s)ds \\
&= \int_{-\pi}^{\pi} 2(x(s)-\overline{x})y'(s)ds \\
&\le \int_{-\pi}^{\pi} (x(s)-\overline{x})^2 ds + \int_{-\pi}^{\pi} y'^2(s)ds \qquad (\text{by } 2ab \le a^2+b^2) \\
&\le \int_{-\pi}^{\pi} x'^2(s)ds + \int_{-\pi}^{\pi} y'^2(s)ds \qquad (\text{by Poincaré's inequality}) \\
&= \int_{-\pi}^{\pi} \big(x'^2(s)+y'^2(s)\big)ds \\
&= 2\pi, \qquad (\text{use } x'^2(s)+y'^2(s)=1)
\end{aligned}$$

whence $A[\gamma] \le \pi$. We have shown that the enclosed area of a simple, closed C^1-curve with perimeter 2π cannot exceed π. As π is the area of the unit circle, the unit circle solves the isoperimetric problem.

Now the uniqueness case. We need to examine the equality signs in our derivation. We observe that the equality holds if and only if $a_n = b_n = 0$ for all $n \ge 2$ in the Fourier series of $x(s)$. So, $x(s) = a_0 + a_1 \cos s + b_1 \sin s$, or

$$x(s) = a_0 + r\cos(s - x_0),$$

where

$$r = \sqrt{a_1^2 + b_1^2}, \ \cos x_0 = \frac{a_1}{r} \ .$$

(Note that $(a_1, b_1) \ne (0,0)$. For if $a_1 = b_1 = 0$, $x(s)$ is constant and $x'^2 + y'^2 = 1$ implies $y'^2(s) = \pm s + b$, and y can never be periodic.) Now we determine y. From the above calculation, when the equality holds,

$$x - \overline{x} - y' = 0.$$

So $y'(s) = x(s) - \overline{x} = r\cos(s - x_0)$, which gives

$$y(s) = r\sin(s - x_0) + c_0 \ , \quad c_0 \text{ constant.}$$

It follows that γ describes a circle of radius r centered at (a_0, c_0). Using the fact that the perimeter is 2π, we conclude that $r = 1$, so the maximum must be a unit circle.

Summarizing, we have the following solution to the classical isoperimetric problem.

Theorem 1.20. *Among all closed, simple C^1-curves of the same perimeter, only the circle encloses the largest area.*

The same proof also produces a dual statement, namely, among all regions which enclose the same area, only the circle has the shortest perimeter.

1.7 Fourier Series and Power Series

Power series and Fourier series are the most common types of series of functions. We would like to make a comparison between these two series.

First of all, a power series is of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k,$$

while a trigonometric series is of the form

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + \sin kx).$$

The convergence of the power series is completely determined by its radius of convergence R which is given by

$$R = \liminf_{k \to \infty} \frac{1}{|a_k|^{1/k}}.$$

It is uniformly convergent on any closed subinterval of $(x_0 - R, x_0 + R)$ and divergent at any point lying outside of this open interval. On the other hand, there is no such simple characterization of convergence for the Fourier series. As a consequence of the M-test, the Fourier series converges uniformly when the coefficients satisfy $a_k, b_k = O(k^{-s})$ for any $s > 1$. In MATH2060 we also learned that for any cosine or sine sequence with decreasing coefficients, the Fourier series converges pointwisely and, in fact, uniformly on any subinterval of $[-\pi, \pi]$ not containing the origin. This follows from an application of the Dirichlet test.

For a function one may associate it with a special series called its Taylor series or a Fourier series. Since the Taylor series is determined by the derivatives of all order of the function at a prescribed point, it is completely determined when the function is defined and smooth in an open interval containing this point. We may say that the Taylor series relies on the local property of the function. On the other hand, the Fourier series is well-defined whenever the function is integrable on $[-\pi, \pi]$. It is global as the Fourier coefficients depend on the integral of the function over $[-\pi, \pi]$. We do not need the function to be differentiable to define the series, let alone smoothness. It shows that Fourier series representation is more feasible than Taylor series expansion.

When it comes to the question of representing a function by its Taylor series or Fourier series, we first recall the Taylor's expansion theorem, under certain regularity conditions which will not be recalled here, we have the formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R$$

where the remainder term R is given by

$$R = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(t - x_0)^n dt.$$

You should compare this formula with (3.1), the corresponding formula for Fourier expansion.

We learned in MATH2060 that not every smooth function is equal to its Taylor series. However, there is a simple way to characterize those functions which do; we call it analytic. In other words, every function f is analytic at x_0 if f is equal to its Taylor series at x_0 in an open interval containing x_0, that is, for some $r > 0$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \quad \forall x \in (x_0 - r, x_0 + r).$$

Observing the power series on the right hand side is in fact convergent for all complex values z, $|z - x_0| < r$, this power series defines a complex analytic function in $\{z \in \mathbb{C} : |z - x_0| < r\}$. We conclude that a smooth function is analytic in an open interval containing x_0 if and only if it is the restriction of a complex analytic function defined locally at x_0. On the other hand, despite the effort of mathematicians of many generations and numerous results, a complete characterization of the class of functions whose Fourier series converge has not settled. Sufficient conditions are discussed in Section 1.3. Of course, when we relax the convergence to L^2-convergence, every function is equal to its Fourier series.

Comments on Chapter 1. Historically, the relation (1.2) comes from a study on the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ denote the displacement of a string at the position-time (x, t). Around 1750, D'Alembert and Euler found that a general solution of this equation is given by

$$f(x - ct) + g(x + ct)$$

where f and g are two arbitrary twice differentiable functions. However, D. Bernoulli found that the solution could be represented by a trigonometric series. These two different ways of representing the solutions led to a dispute among the mathematicians at that time, and it was not settled until Fourier gave many convincing examples of representing functions by trigonometric series in 1822. His motivation came from heat conduction. After that, trigonometric series have been studied extensively and people call it Fourier series in honor of the contribution of Fourier. Nowadays, the study of Fourier series has matured into a branch of mathematics called harmonic analysis. It has equal importance in theoretical and applied mathematics, as well as other branches of natural sciences and engineering.

The book by R.T. Seely, "An Introduction to Fourier Series and Integrals", W.A. Benjamin, New York, 1966, is good for a further reading.

Concerning the convergence of a Fourier series to its function, we point out that an example of a continuous function whose Fourier series diverges at some point can be found in Stein-Sharachi. More examples are available by googling. The classical book by A. Zygmund, "Trigonometric Series" (1959) reprinted in 1993, contains most results before 1960. After 1960, one could not miss to mention Carleson's sensational work in 1966. His result in particular implies that the Fourier series of every function in $R_{2\pi}$ converges to the function itself almost everywhere.

There are several standard proofs of the Weierstrass approximation theorem, among them Rudin's proof in "Principles" by expanding an integral kernel and Bernstein's proof based on binomial expansion are both worth reading. Recently the original proof of Weierstrass by the heat kernel is available on the web. It is nice to take a look too. In Chapter 3 we will reproduce Rudin's proof and then discuss Stone-Weierstrass theorem, a far reaching generalization of Weierstrass approximation theorem.

The elegant proof of the isoperimetric inequality by Fourier series presented here is due to Hurwitz (1859-1919). You may google under "proofs of the isoperimetric inequality" to find several different proofs in the same spirit. The isoperimetric inequality has a higher dimensional version which asserts that the ball has the largest volume among all domains having the same surface area. However, the proof is much more complicated.

The aim of this chapter is to give an introduction to Fourier series. It will serve the purpose if your interest is aroused and now you consider to take our course on Fourier analysis next year. Not expecting a thorough study, I name Stein-Shakarchi as the only reference.